Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Simon Müller
Michele Serra
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Exercises for the course "Linear Algebra I"

Sheet 8

Hand in your solutions on Thursday, 19. Dezember 2019, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 8.1 *(4 points)*

A *polynomial* over a field K in the indeterminate X is of the form $f(X) = \sum_{i=0}^{\infty} a_i X^i$, where $a_i \in K$ for all $i \in \mathbb{N}_0$, and $a_i = 0$ for all but finitely many $i \in \mathbb{N}_0$. Two polynomials $\sum_{i=0}^{\infty} a_i X^i$ and $\sum_{i=0}^{\infty} b_i X^i$ are equal if and only if $a_i = b_i$ for all $i \in \mathbb{N}_0$. If $a_n \neq 0$ for some $n \in \mathbb{N}_0$ and $a_m = 0$ for all $m > n$, we also write $f(X) = \sum_{i=0}^{n} a_i X^i$, and we say that f has *degree* n (Abbreviation: $\deg(f) = n$). The set of all polynomials over K in the indeterminate X is denoted by $K[X]$. We equip $K[X]$ with the addition

$$\sum_{i=0}^{\infty} a_i X^i + \sum_{i=0}^{\infty} b_i X^i := \sum_{i=0}^{\infty} (a_i + b_i) X^i$$

and the scalar multiplication

$$\lambda \left(\sum_{i=0}^{\infty} a_i X^i \right) := \sum_{i=0}^{\infty} (\lambda a_i) X^i \quad (\lambda \in K)$$

You may without proof assume that these operations turn $K[X]$ into a K-vector space.

Now let K be a field and let $d \in \mathbb{N}$ be arbitary. In the following, $K[X]_{\leq d}$ denotes the set of all polynomials over K in the indeterminate X of degree smaller than d or equal to d, and likewise $K[X]_{=d}$ denotes the set of all polynomials over K in the indeterminate X of degree equal to d.

(a) Prove that $K[X]_{\leq d} \cup \{0\}$ is a subspace of $K[X]$. For that purpose show that for all $f, g \in K[X]_{\leq d} \cup \{0\}$ and all $\lambda \in K$ the following holds: $f - g \in K[X]_{\leq d} \cup \{0\}$ and $\lambda f \in K[X]_{\leq d} \cup \{0\}$.

(b) Determine a basis and the dimension of the subspace $K[X]_{\leq d}$ of $K[X]$. What is the dimension of $K[X]$?

(c) Is $K[X]_{=d} \cup \{0\}$ a subspace of $K[X]$?

Exercise 8.2 *(5 points)*

Let $W \subseteq \mathbb{R}^4$ be the subspace generated by

$$\alpha_1 := (1, 2, 2, 1), \quad \alpha_2 := (0, 2, 0, 1), \quad \alpha_3 := (-2, 0, -4, 3).$$

Moreover, consider

$$\alpha_1' := (1, 0, 2, 0), \quad \alpha_2' := (0, 2, 0, 1), \quad \alpha_3' := (0, 0, 0, 3).$$

We define $\mathcal{B} := \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{B}' := \{\alpha_1', \alpha_2', \alpha_3'\}$.

(a) Show that $\mathcal{B}$ and $\mathcal{B}'$ are bases of W.

(b) Let $\beta = (b_1, b_2, b_3, b_4) \in W$ be arbitrary. Compute $[\beta]_{\mathcal{B}}$, i.e. the coordinate column matrix (*Koordinaten-Spaltenmatrix*) of β with respect to the ordered basis $\mathcal{B}$.

(c) Find a matrix $P \in M_{3\times 3}(\mathbb{R})$ such that $[\beta]_{\mathcal{B}} = P[\beta]_{\mathcal{B}'}$.

Exercise 8.3 *(4 points)*

Let K be a field, V a 3-dimensional K-vector space and $\alpha, \beta, \gamma \in V$ linearly independent over K. We consider the **ordered bases**

$$\mathcal{B}_1 := \{\alpha, \beta, \gamma\}, \quad \mathcal{B}_2 := \{\beta, \alpha, \gamma\}, \quad \mathcal{B}_3 := \{\beta, \gamma, \alpha\}.$$

Now let $v \in V$ be arbitrary.

(a) Find a matrix $P \in M_{3\times 3}(K)$ such that $[v]_{\mathcal{B}_1} = P[v]_{\mathcal{B}_2}$.

(b) Find a Matrix $P \in M_{3\times 3}(K)$ such that $[v]_{\mathcal{B}_1} = P[v]_{\mathcal{B}_3}$.

(c) (optional/no points) Find further ordered bases $\mathcal{B}_i$ of V consisting of α, β and γ. How do the respective matrices P look like such that $[v]_{\mathcal{B}_1} = P[v]_{\mathcal{B}_i}$?

Exercise 8.4 *(3 points)*

We consider the 5×5-matrix

$$A := \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in M_{5\times 5}(\mathbb{R}).$$

Determine a basis and the dimension of the row space W of A.